Semi-Markov Modeling of Standby Systems with N components and N Redundant units

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Abstract

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Consider a system which has an operating series subsystem consisting of N components, an identical stand-by subsystem and a replacing switch. Also, suppose that a technician is present on the site to repair or replace failed elements of subsystems in the event of a breakdown, each subsystem consists of N components. Grabski (2010), obtained a tedious closed form for calculating the Reliability parameter of the above system in cold standby configuration. Because of the complicated forms of the introduced formulas, they are not applicable in practice. In this article an approximation technique and some simulation study is done for reliability analysis of certain similar system.

keywords: Semi-Markov, Redundancy, Cold Standby, Reliability.

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Introduction

Reliability function is a fundamental factor which plays an important role in evaluating the performance of engineering systems. To increase the reliability of a system, there are a lot of approaches, one of those is using redundant components in parallel. There are two well-known types of redundancy strategies: active and standby. In active redundancy, all components begin to operate simultaneously at time zero, whereas in standby redundancy, redundant components are sequentially put into operation whenever an active one fails.

Since in standby redundancy(cold standby), the redundant component(system) has the failure rate of zero value, more attention is paid to this method. In this study, a repairable cold standby system is considered and its reliability is obtained by using the semi-Markov modelling.

Description and Construction the Semi-Markov Model

Suppose that the system consists of an operating series subsystem, an identical stand-by subsystem and a switch. Also, Suppose that a technician is on site to repair or replace failed elements of subsystems in the event of a breakdown.
Each subsystem consists of N components.

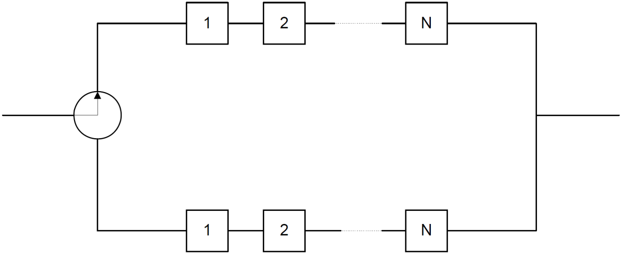

Figure 1: Diagram of the system

When the operating subsystem fails, the switch acts and replaces the standby subsystem with the operating one. Suppose that U is the random variable corresponding to the operation of switch with Bernouli distribution:

$$b(k) = P(\mathrm{U} = k) = p^k (1-p)^{1-k} \qquad k = 0, 1. \qquad 0 < p < 1. \quad (2.1)$$

System fails whenever the operating subsystem fails and the previously failed subsystem has not been still renewed, or when the operating subsystem fails and the replacing switch fails, also. Assume that by failure of the whole system, it is replaced by a new identical one. The time to replace the new system is nonnegative random variables η with CDF

$$K(x) = P(\eta \le x), \qquad x \ge 0. \quad (2.2)$$

Suppose that distributions of the times to failure of elements are represented by non-negative mutually independent random variables

$$\zeta_k, \quad k = 1, ..., N,$$

with probability density functions $f_k(x)$, $x \geq 0$, $k = 1, ..., N$. also, assume that the lengths of repair periods of failed units are represented by identical copies of non-negative random variables γ_k, $k = 1, ..., N$, with cumulative distribution functions:

$$H_k(x) = P(\gamma_k \leq x), \qquad x \geq 0.$$

Moreover, we assume that all above mentioned random variables to be independent.

Constructing a Semi-Markov model

Consider the following states:

0 : Failure of the whole system;

k : Renewal of the failed subsystem after the failure of k^{th} component, $k = 1, ..., N$, and operation of the spare unit;

$N + 1$: All operating units and the corresponding spares are "up".

Let
$0 = \tau_0^*, \tau_1^*, \tau_2^*, ...$ - denote the instants of the states changes,

$\{Y(t) : t \geq 0\}$ be a random process on the $S = \{0, 1, ..., N, N + 1\}$,

which keeps constant on the half-intervals $[\tau_n^*, \tau_{n+1}^*)$; $n = 0, 1, ...$

Create a new process by this way:

Let τ_0 and , $\tau_1, \tau_2, ...$ denote the instants of the subsystem failures or

instants of the whole system renewal. The random process $X(t) : t \geq 0$ defined by

$$X(0) = 0, \quad X(t) = Y(\tau_n), \qquad \text{for } t \in [\tau_n, \tau_{n+1}) \qquad (2.3)$$

is a SMP describing the system.

Semi-Markov Kernel

The semi-Markov kernel is:

$$\mathrm{Q}(t) = [\mathrm{Q}_{ij}(t) : \quad i, j \in S] \tag{3.1}$$

where

$$Q_{ij}(t) = P(\tau_{n+1} - \tau_n \leq t, X(\tau_{n+1}) = j | X(\tau_n) = i), \qquad t \geq 0 \tag{3.2}$$

The sequence $\{X(\tau_n) : n = 0, 1, ...\}$ is called the embedded Markov chain with transition matrix $P = [p_{ij} = Q_{ij}(\infty) : \quad i, j \in]$.

$$p_{ij} = \lim_{t \to \infty} \mathrm{Q}_{ij}(t) = P(X(\tau_{n+1}) = j | X(\tau_n) = i) \tag{3.3}$$

T_i : the waiting time of state i when the next state is unknown.
The cdf of random variable T_i is:

$$G_i(t) = \sum_{i \in S} \mathrm{Q}_{ij}(t) = P(\tau_{n+1} - \tau_n \leq t | X(\tau_n) = i) \tag{3.4}$$

T_{ij} : the waiting time of state i when the next stat is j.
The cdf of random variable T_{ij} is:

$$F_{ij}(t) = P(\tau_{n+1} - \tau_n \leq t | X(\tau_n) = i, X(\tau_{n+1}) = j) = \frac{\mathrm{Q}_{ij}(t)}{p_{ij}} \tag{3.5}$$

The semi-Markov kernel has the form:

$$\mathbf{Q}(t) = \begin{bmatrix} 0 & 0 & \cdots & 0 & Q_{0,N+1}(t) \\ Q_{1,0}(t) & Q_{11}(t) & \cdots & Q_{1,N}(t) & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Q_{N+1,0}(t) & Q_{N+1,1}(t) & \cdots & Q_{N+1,N}(t) & 0 \end{bmatrix}$$

where after computations its elements become as follows:

$$Q_{N+1j}(t) = p\int_0^t \prod_{i\neq j}^{N}[1-F_i(x)]f_j(x)dx, \qquad j = 1, ..., N$$

$$Q_{N+10}(t) = (1-p)(1-\prod_{i=1}^{N}[1-F_i(x)]), \qquad j = 0$$

$$Q_{ij}(t) = p\int_0^t H_i(x)\prod_{k\neq j}^{N}[1-F_k(x)]f_j(x)dx, \qquad i,j = 1, ..., N$$

$$Q_{i0}(t) = p\int_0^t H_i(x)dF(x), \qquad i = 1, ..., N$$

where

$$F(x) = 1-\prod_{k=1}^{N}[1-F_k(x)]$$

From the assumption it follows that

$$Q_{0N+1}(t) = K(t)$$

Example: Consider a 3-state semi-markov process with the following kernel

$$\mathbf{Q(t)} = \begin{bmatrix} 0 & 0 & Q_{02}(t) \\ Q_{10}(t) & Q_{11}(t) & 0 \\ Q_{20}(t) & Q_{21}(t) & 0 \end{bmatrix} \tag{3.6}$$

where

$$\mathrm{Q}_{11}(t) = p\int_0^t H(x)dF(x) \qquad \mathrm{Q}_{10}(t) = F(t) - p\int_0^t H(x)dF(x)$$

$$\mathrm{Q}_{20}(t) = (1-p)F(t) \qquad \mathrm{Q}_{21}(t) = pF(t)$$

and

$$\mathrm{Q}_{02}(t) = K(t)$$

Assume that, the initial state is 2 . It means that an initial distribution is

$$p(0) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \tag{3.7}$$

An Approximate Reliability Function

In this section an approximate reliability function of the system by using results from the theory of semi-Markov processes perturbations has been presented.

The first arrival at the set of states $A \subset S$ of the embedded Markov chain $\{X(\tau_n) : n \in N_0\}$ is difined as

$$\Delta_A = min\{n \in N : X(\tau_n) \in A\} \quad (4.1)$$

The first passage time to the set of states $A \subset S$ of the semi-Markov process $\{X(t) : t \geq 0\}$ is denoted by

$$\Theta_A = \tau_{\Delta_A} \quad (4.2)$$

Suppose that Θ_{iA} denotes the first passage time from the state $i \in \acute{A}$ to a subset A. The CDF of Θ_{iA} is the function

$$\Phi_{iA}(t) = P(\Theta_A \leq t | X(0) = i), \quad t \geq 0 \quad (4.3)$$

Theorem 1. For the regular semi-Markov processes such that,

$$f_{iA} = P(\Delta_A < \infty | X(0) = i) = 1, \quad i \in \acute{A} \tag{4.4}$$

distributions are proper and they are the unique solutions of the equations system

$$\Phi_{iA}(t) = \sum_{j \in A} \mathrm{Q}_{ij}(t) + \sum_{k \in S} \int_0^t \Phi_{kA}(t-x) d\mathrm{Q}_{ik}(x), i \in \acute{A} \bullet \tag{4.5}$$

Theorem 2. If $f_{iA} = 1$ and $E(T_{ij}^2)$ be bounded then there exist expectations $E(\theta_{iA}), i \in \acute{A}$ and $E(\theta_{iA}^2), i \in \acute{A}$ and they are unique solutions of the linear systems equations, which have following matrix forms

$$(\mathbf{I} - \mathbf{P}_{\tilde{A}})\bar{\Theta}_{\tilde{A}} = \bar{\mathbf{T}}_{\tilde{A}} \tag{4.6}$$

where

$$\mathbf{P}_{\tilde{A}} = [p_{ij} : i, j \in \acute{A}], \quad \bar{\Theta}_{\tilde{A}} = [E(\Theta_{iA}) : i \in \acute{A}]^T, \quad \bar{\mathbf{T}}_{\acute{A}} = [E(T_i) : i \in \tilde{A}]$$

$$(\mathbf{I} - \mathbf{P}_{\acute{A}})\bar{\Theta^2}_{\acute{A}} = \bar{\mathbf{B}}_{\acute{A}} \tag{4.7}$$

where

$$\mathbf{P}_{\acute{A}} = [p_{ij} : i, j \in \acute{A}], \quad \bar{\Theta^2}_{\acute{A}} = [E(\Theta_{iA}^2) : i \in \acute{A}]^T$$

$$, \bar{\mathbf{B}}_{\acute{A}} = [b_{iA} : i \in \acute{A}]^T, \quad b_{iA} = E(T_i^2) + 2 \sum_{i \in \acute{A}} p_{ik} E(T_{ik}) E(\theta_{iA})$$

and $\mathbf{I}$ is the unit matrix. $\bullet$

Here the time to failure of the system is denoted θ_{iA} that is the first passage time from the state i=2 to the subset $A = \{0\}$. Therefore, the reliability function

$$R(t) = P(\theta_{20} > t) = 1 - \Phi_{20}(t), t \geq 0. \tag{4.8}$$

The transition matrix of the embedded Markov chain of the semi-Markov process $\{X(t) : t \geq 0\}$ is

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 \\ p_{10} & p_{11} & 0 \\ p_{20} & p_{21}(t) & 0 \end{bmatrix} \tag{4.9}$$

where

$$p_{10} = 1 - p_{11}, \quad p_{11} = p\int_0^{\infty} H(x)dF(x), \quad p_{20} = 1 - p, \quad p_{21} = p$$

The CDF of the waiting times $T_i, i = 0, 1, 2$ are

$$G_0(t) = K(t), \quad G_1(t) = F(t), \quad G_2(t) = f(t)$$

Hence

$$E(T_0) = E(\eta), \quad E(T_1) = E(\zeta), \quad E(T_2) = E(\zeta) \tag{4.10}$$

In this case the solution of equation (4.7) is:

$$E(\Theta_{10}) = \frac{E(\zeta)}{1 - p_{11}} \qquad E(\Theta_{20}) = E(\zeta) + \frac{pE(\zeta)}{1 - p_{11}} \tag{4.11}$$

Let $\acute{A} = S - A$ be a finite subset of states and A be at least countable subset of S. Suppose $\{X(t) : t \geq 0\}$ is SM process with the state space $S = A \cup \acute{A}$ and the kernel $\mathbf{Q}(t) = [\mathrm{Q}_{ij} : i, j \in S]$, the elements of which have the form $\mathrm{Q}_{ij} = p_{ij} F_{ij}(t)$. Assume that

$$\epsilon_i = \sum_{j \in A} p_{ij}, \qquad p^0_{ij} = \frac{p_{ij}}{1 - \epsilon_i}, \quad i, j \in \acute{A}$$

A semi-Markov process$\{X(t) : t \geq 0\}$ with the discrete state space S defined by the renewal kernel $\mathbf{Q}(t) = [p_{ij} F_{ij}(t) : i, j \in S]$, is called the perturbed process with respect to SM process $\{X^0(t) : t \geq 0\}$ with the state space $\acute{A}$ defined by the kernel $\mathbf{Q}^0(t) = [p^0_{ij} F_{ij}(t) : i, j \in \acute{A}]$.

The number

$$m_i^0 = \int_0^\infty [1 - G_i^0(t)]dt, \qquad i \in \acute{A} \tag{4.12}$$

where

$$G_i^0(t) = \sum_{j \in \acute{A}} Q_{ij}^0(t) \tag{4.13}$$

is the expected value of the waiting time in state for the process $\{X^0(t) : t \geq 0\}$.
Denote the stationary distribution of the embedded Markov chain in SM process $\{X^0(t) : t \geq 0\}$ by $\pi^0 = [\pi_i^0 : i \in \acute{(A)}]$. Let

$$\epsilon = \sum_{i \in \acute{A}} \pi_i^0 \epsilon_i, \qquad m^0 = \sum_{i \in \acute{A}} \pi_i^0 m_i \tag{4.14}$$

let $\Theta_{iA} = inf\{t : X(t) \in A | X(0) = i\}, i \in \acute{A}$

Theorem 3. If the embedded Markov chain defined by the matrix of transition probabilities $P = [p_{ij} : i, j \in S]$ satisfies the following conditions

$$f_{iA} = P(\Delta_A < \infty | X(0) = i) = 1, \qquad i \in \acute{A}$$

$$\exists c > 0 \quad \text{such that} \quad \forall i, j \in S \quad 0 < E(T_{ij}) < c$$

then

$$\lim_{\epsilon \to 0} P\Theta_{iA} > x) = e^{-\frac{\epsilon}{m^0} x} \qquad \bullet \qquad (4.15)$$

In the following a simulation study for reliability of the semi-markov process with three states has been done.

Simulation

Example 1. Suppose that probability of the switch works is 0.8, distribution of that the lengths of repair periods is $E(4)$, distribution of time to failure of elements is $Gamma(3, 10)$ and distribution of replacing time of system is $E(1)$.

The transition matrix of the embedded Markov chain of the semi-Markov process is

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 \\ 0.21 & 0.79 & 0 \\ 0.2 & 0.8 & 0 \end{bmatrix}$$

The reliability function is

$$R(t) = e^{-\frac{0.21}{3.01}t}$$

that is shown in Figure 2.

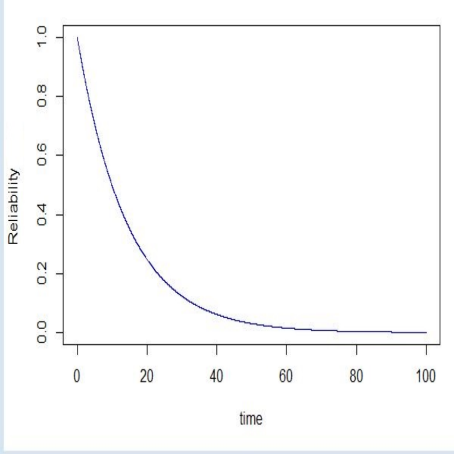

Figure: Figure 2: Reliability function with gamma lifetime distribution

Figure 3 compare two reliability functions, one is calculated using analytical formulas and the other is simulated with numerical values.

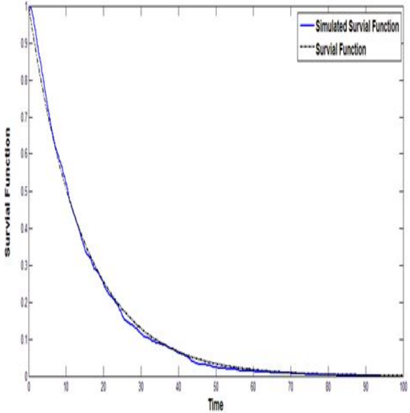

Figure: Figure 3: Comparison of simulated and analytical Reliability functions

Figures 4 and 5 show the distribution and the failure rate function of the simulated data:

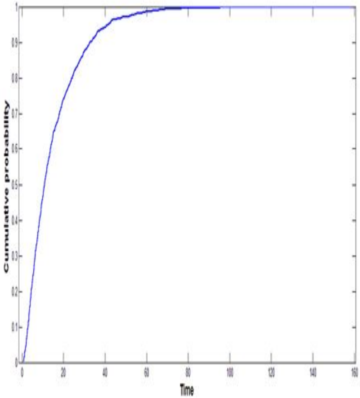

Figure: Figure 4: Distribution function for lifetime data

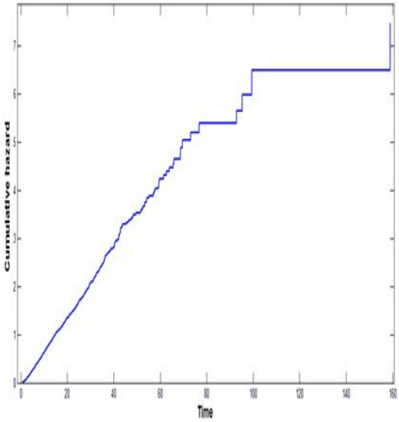

Figure: Figure 5: Distribution function for lifetime data

Example 2. Suppose that probability of the switch works is 0.95, distribution of that the lengths of repair periods is $W(3,1)$, distribution of time to failure of elements is $E(1)$ and distribution of replacing time of system is $E(2)$. The transition matrix of the embedded Markov chain of the semi-Markov process is

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 \\ 0.5904 & 0.4096 & 0 \\ 0.05 & 0.95 & 0 \end{bmatrix} \tag{5.1}$$

The reliability function is as follows

$$R(t) = e^{-\frac{0.5904}{1.791}} \tag{5.2}$$

Figure 6 shows of this function

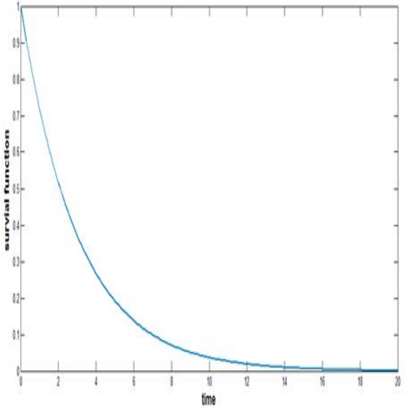

Figure: Reliability function with Exponential lifetime distribution

Figure 7 compare two reliability functions, one is calculated using analytical formulas and the other is simulated with numerical values.

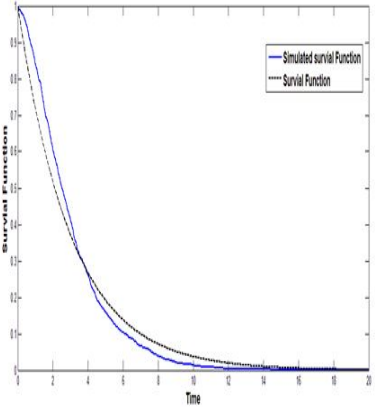

Figure: Comparison of simulated and analytical Reliability functions

Figures 8 and 9 show the distribution and the failure rate function of the simulated data:

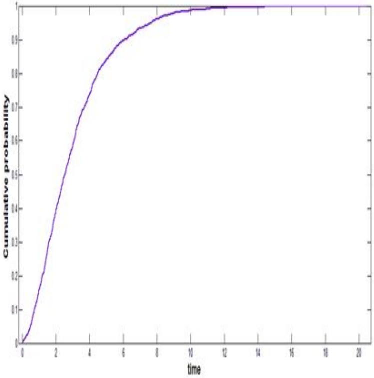

Figure: Distribution function for lifetime data

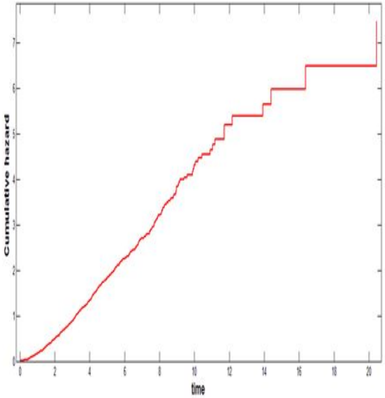

Figure: Distribution function for lifetime data

conclusion

The approximation of reliability function for semi-Morkov process defining on repairable cold standby system is obtained. Also, Figures 3 and 7 show that this function is good agreement with the reliability function obtained in other ways (calculated by the software MATLAB in a different way) for the simulated data.

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