A review On Robust Asymptotic Theory of Unstable AR(p) Processes with Infinite Variance

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Outline

1. Introduction
   - Stable Distribution
   - AR Model
   - M-estimate Method
2. The Limiting Distribution for AR(p)
   - Simulation Study
3. AR Models with a Location Parameter
   - Mean Estimation
     - Robust Estimator for the Mean Vector
   - Simulation Study

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Introduction

Stable Distribution

The distribution S is stable if for $X, X_1, X_2, \ldots \overset{i.i.d.}{\sim} S$,

$$\exists\ c_n > 0, \lambda_n \in \mathbb{R} \quad \ni \quad \sum_{i=1}^{n} X_i \overset{d}{=} c_n X + \lambda_n.$$

Domain of Attraction of a Stable Law

The distribution $F \in DS(\alpha)$ if for $X_1, X_2, \ldots \overset{i.i.d.}{\sim} F$,

$$\exists\ a_n > 0, b_n \in \mathbb{R} \quad \ni \quad a_n^{-1} \left(\sum_{i=1}^{n} X_i - b_n \right) \overset{d}{\to} S,$$

where S has a stable distribution with index $0 < \alpha \leq 2$.

Introduction

AR Model

The AR(p) model is defined to be

$$\phi(B)X_t = \epsilon_t, \tag{1}$$

where B is backward operator and

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p.$$

Here, we assume that

$$\epsilon_t \overset{i.i.d.}{\sim} F, F \in DS(\alpha), \tag{2}$$

Introduction

AR(1) Model

The AR(1) model is defined to be

$$X_t = \phi X_{t-1} + \epsilon_t,\ t = 2, 3, \ldots, n,\ X_0 = 0 \tag{3}$$

If $|\phi| < 1$ and $F \in DS(2)$ then

$$\sqrt{n}(\hat{\phi} - \phi) \xrightarrow{d} N(0, 1 - \phi^2)$$

and when $\phi = \pm 1$

$$n(\hat{\phi} - 1) \xrightarrow{d} U$$

and U is not Gaussian (result from Anderson). This is also called Dickey–Fuller unit root test. In here

$$\hat{\phi} = \frac{\sum_{t=2}^{n} X_t X_{t-1}}{\sum_{t=2}^{n} X_{t-1}^2}.$$

Introduction

For stock prices if $e^{-rt}S_t = Y_t$ and random walk model is based on

$$\log \frac{Y_t}{Y_{t-1}} = \epsilon_t$$

is equivalent to

$$\log Y_t = \log Y_{t-1} + \epsilon_t$$

is random walk model where $X_t = \log Y_t$.

M-estimate Method

Consider the general AR(p) model. The classical M-estimate, $\hat{\Phi} = (\hat{\phi}_1, \hat{\phi}_2, \ldots, \hat{\phi}_p)^T$, of $\Phi = (\phi_1, \phi_2, \ldots, \phi_p)^T$ minimizes

$$g(\beta_1, \cdots, \beta_p) = \sum_{t=p+1}^{n} \rho(X_t - \beta_1 X_{t-1} - \cdots - \beta_p X_{t-p}),$$

with respect to $(\beta_1, \ldots, \beta_p)$.

Assumptions on the function $\rho(\cdot)$:

A1. let ρ be a convex and twice differentiable function, and take $\psi = \rho'$.

A2. $\mathbb{E}(\psi(\epsilon_1)) = 0$ and $\mathbb{E}(\psi^2(\epsilon_1)) < \infty$.

A3. $0 < |\mathbb{E}(\psi'(\epsilon_1))| < \infty$ and $\psi'(\cdot)$ satisfies the Lipschitz-continuity condition.

AR(1) Model with unit root

The AR(1) model is defined to be

$$X_t = \phi X_{t-1} + \epsilon_t, \tag{4}$$

where $\phi = \pm 1$. Here, we assume that

$$\epsilon_t \overset{i.i.d.}{\sim} F, F \in DS(\alpha), \tag{5}$$

The goal is to estimate ϕ to test the unit root hypothesis is valid. W.L.O.G. we only consider $\phi = 1$.

Introduction

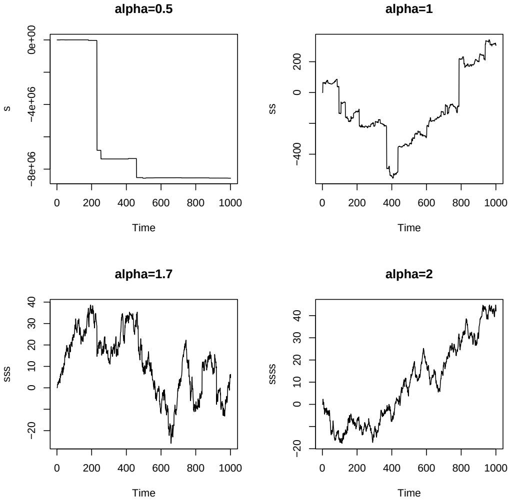
Partial Sum Process

If $\epsilon_i \overset{i.i.d.}{\sim} F$ and F is in the domain of attraction of a stable law with index $\alpha \in (0, 2]$ then

$$a_n^{-1} \sum_{i=1}^{[nt]} (\epsilon_i - c_n) \xrightarrow{d} S(t)$$

where S_t is the stable process with index $(0, 2]$.

For $\alpha \in (0, 1)$ we can take $c_n = 0$ and for $\alpha \in (1, 2]$ when $\epsilon_i \overset{d}{=} -\epsilon_i$, we can take $c_n = 0$. When $\alpha = 2$, the process S is Brownian motion.


alpha=0.5
alpha=1
alpha=1.7
alpha=2
s
ss
sss
ssss
Time
0e+00
−4e+06
−8e+06
200
0
−400
40
30
20
10
0
−20
0 200 400 600 800 1000

Asymototic theory for $\hat{\phi}_M$

We have

$$a_n n^{1/2}(\hat{\phi}_M - 1) \xrightarrow{d} \frac{E^{1/2}(\psi^2(\epsilon_1)) \int_0^1 S(s)\, dW(s)}{E(\psi'(\epsilon_1)) \int_0^1 S^2(s)\, ds}$$

and

$$\left(\sum_{t=2}^{n} X_{t-1}^2 \right)^{1/2} (\hat{\phi}_M - 1) \xrightarrow{d} \frac{E^{1/2}(\psi^2(\epsilon_1))}{E(\psi'(\epsilon_1))}\, W(1).$$

Here S is the symmetric stable law with index $\alpha \in (0, 2]$ and W is Brownian motion and $a_n/\sqrt{n} \to \infty$. This shows that M-estimates derive higher rate of convergance (better estimate of ϕ). (**Knight (1989)**)

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The Limiting Distribution for AR(p)

Characteristic polynomial of AR(p):

$$\phi(z) = (1-z)^r (1+z)^s \prod_{k=1}^{l} (1 - 2\cos(\theta_k) z + z^2)^{d_k} \varphi(z).$$

Note that the roots of equation

$$1 - 2\cos(\theta) z + z^2 = 0$$

are $z_0 = \exp(i\theta)$ and $\bar{z}_0 = \exp(-i\theta)$ and $||z_0|| = ||\bar{z}_0|| = 1$.

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are $z_0 = \exp(i\theta)$ and $\bar{z}_0 = \exp(-i\theta)$ and $||z_0|| = ||\bar{z}_0|| = 1$. There exists a nonsingular $p \times p$ matrix Q (Chan and Wei (1988)) such that

$$Q\mathbf{X_t} = \left(\mathbf{u}_t^T, \mathbf{v}_t^T, \mathbf{w}_t^T(1), \ldots, \mathbf{w}_t^T(l)\right)^T,$$

where we assume that $\varphi(z) = 1$ and

$$\mathbf{X_t} = (X_t, \ldots, X_{t-p+1})^T,$$

$$\mathbf{u_t} = (u_t, \ldots, u_{t-r+1})^T,\ u_t = \phi(B)(1-B)^{-r} X_t,$$

$$\mathbf{v_t} = (v_t, \ldots, v_{t-s+1})^T,\ v_t = \phi(B)(1+B)^{-s} X_t,$$

$$\mathbf{w_t}(k) = (w_t(k), \ldots, w_{t-2d_k+1}(k))^T,$$

The Limiting Distribution for AR(p)

Moreover, let

$$G_n = \operatorname{diag}\left(J_n, K_n, L_n(1), \ldots, L_n(l)\right).$$

Then, we have

$$G_n Q \mathbf{X_t} \;=\; \operatorname{diag}\left(J_n \mathbf{u_t}, K_n \mathbf{v_t}, L_n(1)\mathbf{w_t}(1), \ldots, L_n(l)\mathbf{w_t}(l)\right) + o_p(1).$$

The Limiting Distribution for AR(p)

Theorem 1

Suppose $\{X_t\}$ satisfies (4) and conditions A1-A3 hold. Then

$$\left(Q^T G_n^T\right)^{-1}(\hat{\Phi} - \Phi) \xrightarrow{d} \left((\Gamma^{-1}\mathcal{F})^T, (\Upsilon^{-1}\mathcal{H})^T, (\Lambda_1^{-1}\mathcal{G}_1)^T, \ldots, (\Lambda_l^{-1}\mathcal{G}_l)^T\right)^T,$$

where $(\Gamma^{-1}\mathcal{F})$, $(\Upsilon^{-1}\mathcal{H})$, and $(\Lambda_i^{-1}\mathcal{G}_i)$ for $i = 1, \ldots, l$ are pretty complex and are defined in the paper.

Simulation Study

Consider the model $X_t = (2\cos\theta)X_{t-1} - X_{t-2} + \epsilon_t$. This is in fact a deterministic process

$Y_t = A\cos(\omega t) + B\sin(\omega t)$, A independent from B and ω is a constant,

added with noise.

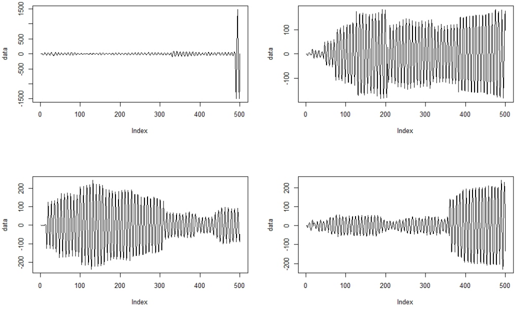

Figure 1: Different sample paths for this model when $\alpha = 1.3$ and $n = 500$.

Simulation Study

Table 1: Median and 90% IPR (in parentheses) for $|\hat{\phi}_1 - 2\cos\theta|$ in model (**??**) by the M-estimate method using the Huber loss function

	Index of stability α			
n	0.5	1	1.7	2
10	0.0258(0.4141)	0.1170(0.4097)	0.1839(0.4003)	0.1947(0.3961)
30	0.0009(0.0245)	0.0180(0.1367)	0.0444(0.2133)	0.0538(0.2323)
50	0.0002(0.0063)	0.0083(0.0584)	0.0263(0.1278)	0.0318(0.1396)

Table 2: Median and 90% IPR (in parentheses) for $|\hat{\phi}_1 - 2\cos\theta|$ in model (**??**) by the LS estimate method

	Index of stability α			
n	0.5	1.0	1.7	2.0
10	0.0806(1.6076)	0.1351(0.7979)	0.1858(0.6077)	0.1992(0.5757)
30	0.0185(0.1748)	0.0349(0.1742)	0.0482(0.1788)	0.0526(0.1879)
50	0.0106(0.0964)	0.0202(0.1056)	0.0290(0.1078)	0.0312(0.1138)

$$\left(Q^T G_m^T\right)^{-1}(\hat{\Phi}^* - \hat{\Phi}) \xrightarrow{d} \left((\Gamma^{-1}\mathcal{F})^T, (\Upsilon^{-1}\mathcal{H})^T, (\Lambda_1^{-1}\mathcal{G}_1)^T, \ldots, (\Lambda_l^{-1}\mathcal{G}_l)^T\right)^T,$$

in probability.

Table 3: Coverage for the naive 95% bootstrap confidence interval for ϕ_1 in model $X_t = 2\cos\theta X_{t-1} - X_{t-2} + \epsilon_t$

α	1.3			1.7		
n	50	100	200	50	100	200
$m = n/\ln(\ln(n))$	96.1%	96.5%	97.4%	95.1%	96.5%	96.8%
$m = n^{(0.9)}$	97.2%	97.4%	97.7%	96.6%	96.9%	97.4%
$m = n^{(0.95)}$	95.7%	96.6%	96.3%	94.0%	94.8%	94.8%

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AR Models with a Location Parameter

Consider the following AR(p) process

$$X_t = \mu + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \epsilon_t, \quad t = 1, 2, \ldots, n, \qquad (6)$$

where $\{\epsilon_t\} \in DS(\alpha)$ and μ is the location parameter (mean when $1 < \alpha \leq 2$).

Mean Estimation

Consider the model

$$\mathbf{X}_i = \boldsymbol{\mu} + \boldsymbol{\epsilon}_i,\ i = 1, \ldots, n, \tag{7}$$

where

- $\mathbf{X}_i = (X_{i1}, \ldots, X_{ip})'$, $i = 1, \ldots, n$
- $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_p)'$
- $\{\boldsymbol{\epsilon}_i\} = \{(\epsilon_{i1}, \ldots, \epsilon_{ip})'\}$ form sequences of i.i.d. random vectors with zero mean in $DS(\alpha_1, \ldots, \alpha_p)$ where $\alpha_j \in (1, 2]$, for $j = 1, \ldots, p$.

Motivation

- $\bar{\mathbf{X}}_n \to \boldsymbol{\mu}$ with the rate of convergence na_n^{-1} where $a_n = n^{-1/\alpha}L(n)$ and L is a slowly varying functions at ∞.

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- *Robust* estimator for the Mean Vector: the M-estimate, $\hat{\boldsymbol{\mu}}_M$, of $\boldsymbol{\mu}$ minimizes

$$\sum_{i=1}^{n} \left(\rho\left(\mathbf{X}_i - \boldsymbol{\mu}\right) - \rho\left(\boldsymbol{\epsilon}_i\right)\right),$$

where errors are in DS with possibly different indices of stability in $(1, 2]$.

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  where errors are in DS with possibly different indices of stability in $(1, 2]$.

For the multivariate loss function use

$$\rho\left(x_1, \ldots, x_p\right) = \rho_1\left(x_1\right) + \cdots + \rho_p\left(x_p\right).$$

Robust Estimator for the Mean Vector

Theorem 2

Suppose (7) holds. Let $\hat{\boldsymbol{\mu}}_M$ be the M-estimator of the mean vector for a sequence of i.i.d. observations in the domain of attraction of a stable law with indices of stability $(\alpha_1, \ldots, \alpha_p)$ such that $1 < \alpha_j \leq 2$, $j = 1, \ldots, p$. Then, we have

$$\mathbf{W}_n = \sqrt{n}(\hat{\boldsymbol{\mu}}_M - \boldsymbol{\mu}) \xrightarrow{d} \mathbf{W}, \tag{8}$$

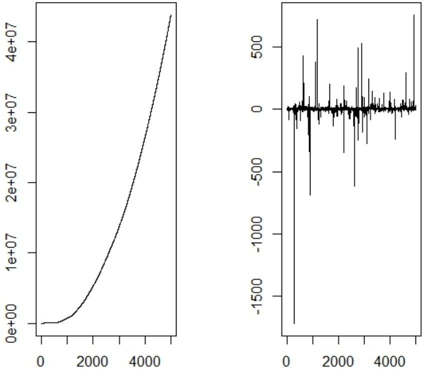
where $\mathbf{W}$ has a multivariate normal distribution with mean zero and covariance matrix $\Sigma = \text{diag}\left(\frac{\mathbb{E}[(\psi_1(\epsilon_{11}))^2]}{\mathbb{E}^2(\psi_1'(\epsilon_{11}))}, \ldots, \frac{\mathbb{E}[(\psi_\text{p}(\epsilon_{1\text{p}}))^2]}{\mathbb{E}^2(\psi_\text{p}'(\epsilon_{1\text{p}}))}\right)$.

Remark: $na_n^{-1} \leq \sqrt{n}$.

M-estimates of the parameters in AR(2) with a Location Parameter

Consider the model

$$X_t = 4 + 2X_{t-1} - X_{t-2} + \epsilon_t. \tag{9}$$



Xt = 4 + 2X_t-1- X_t-2 + epsilon_t   Differencing the time series 2 times

Table 4: M-estimates of the parameters in model (9) with sample size $n = 100$ and replication size $10,000$

	$n = 20$			$n = 100$		
α	ϕ_1	ϕ_2	μ	ϕ_1	ϕ_2	μ
0.5	1.9811	-0.9796	4.2711	1.9999	-0.9999	4.0105
1.0	1.9588	-0.9552	4.5645	1.9983	-0.9982	4.1414
1.5	1.9635	-0.9596	4.5279	1.9982	-0.9981	4.1504
1.8	1.9716	-0.9686	4.4169	1.9986	-0.9986	4.1122
2.0	1.9759	-0.9733	4.3496	1.9989	-0.9989	4.0934

Thanks!

Some extras

- In the stationary time series if $\{\epsilon_t\}$ are i.i.d. with all moments (Mann and Wald (1943))

$$\sqrt{n}(\hat{\Phi}_{LS} - \Phi) \xrightarrow{d} N(0, \Sigma).$$

- For random walk Model:
  - when $\{\epsilon_t\} \sim N(0, \sigma^2)$ (White (1985))

$$n(\hat{\phi}_{LS} - 1) \xrightarrow{d} \tau = \frac{W^2(1) - 1}{2\int_0^1 W^2(s)ds},$$

    where $W(\cdot)$ is a standard Brownian-motion process.
  - when $\{\epsilon_t\} \in DS(\alpha)$ (Chan and Tran (1989))

$$n(\hat{\phi}_{LS} - 1) \xrightarrow{d} \frac{S^2(1) - V(1)}{2\int_0^1 S^2(s)ds},$$

    where $S(\cdot)$ and $V(\cdot)$ are stable processes. Knight (1989) proves that

$$n^{1/2}a_n(\hat{\phi}_M - 1) \xrightarrow{d} \frac{E^{\frac{1}{2}}\left(\psi^2(\epsilon_1)\right)}{E\left(\psi'(\epsilon_1)\right)} \frac{\int_0^1 S(t)dW(t)}{\int_0^1 S^2(t)dt}$$

- If $\{\epsilon_t\} \in DS(\alpha)$ and stationary AR model (Davis, Knight Liu (1992))

$$a_n(\hat{\Phi}_M - \Phi) \xrightarrow{d} \text{to some random vector } \boldsymbol{\xi}_1$$

- Unstable AR(2) model with double root 1 (Chan and Zhang (2012))

$$\begin{pmatrix} n(\hat{\phi}_{LS1} - 2) \\ n^2(\hat{\phi}_{LS1} - 2) + n^2(\hat{\phi}_{LS2} + 1) \end{pmatrix} \xrightarrow{d} \text{to some random vector } \boldsymbol{\xi}_2$$

- Unstable AR(2) model with double root 1 by M-estimate method

$$\begin{pmatrix} n^{1/2}a_n(\hat{\phi}_{M1} - 2) \\ n^{3/2}a_n(\hat{\phi}_{M1} - 2) + n^{3/2}a_n(\hat{\phi}_{M2} + 1) \end{pmatrix} \xrightarrow{d} \text{to some random vector } \boldsymbol{\xi}_3$$