Remarks on Partial Stochastic Differential Equations

L. Gawarecki

Kettering University

To my mentor Professor V. Mandrekar

Let's chat about...

Hello

- I met Professor Mandrekar at Michigan State in 1989, he was a caring mentor with a vibrant research program
- We collaborated for many productive and highly fulfilling years, written joint papers, presented at conferences, and published two books
- Mandrekar had broad research interests: measure and ergodic theory, prediction, vector and operator-valued measures, probability in Banach spaces, stochastic analysis, filtering, control, Wiener-Itô multiple integrals, random fields, Gaussian processes, statistics
- Participated in more than 40 International Conferences as an invited speaker at prestigious places such as Strasbourg, Oberwolfach, Banach Center, Trinity College, Johns Hopkins, Nagoya, IMA, Math. Sci. Inst. (Cornell), CIMAT (Mexico), ISI (Delhi, Calcutta), NATO- ASI, LUCAC Conference, IFIP and All India Science Congress.

Skorokhod, Ogawa, and Itô-Ramer Integrals

- Extension of Ogawa integral; Ogawa integral with respect to Gaussian processes and its relationship to the Skorokhod integral.
- Extension of Itô-Ramer integral to Gaussian processes and its relationship to the Skorokhod integral.
- Anticipative stochastic differential equations

Infinite Dimensional Equations

Two fundamental Problems:

- Peano's Theorem is false in infinite dimensional Banach spaces

Theorem (Peano)

For every continuous function $f : \mathbb{R} \times B \to B$ *defined on an open set* $V \subset \mathbb{R} \times B$ *and for every point* $(t_0, x_0) \in V$ *the Cauchy problem*

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0$$

has a solution defined on some neighbourhood of t_0.

- The problem is with compactness. Bounded sets in $\mathbb{R}$ are pre-compact, but a closed unit ball in an infinite dimensional Banach space in not compact.

Theorem (Godunov, 1973)

Every Banach space, where Peano's theorem is true is finite dimensional.

- The presence of unbounded operators in differential equations:
  The Laplacian

$$\Delta : W^{1,2} \to W^{1,2}$$

  on the Sobolev space (completion of the space of functions with continuous and square integrable derivatives in the Sobolev L_2 norm) is unbounded.
- First, there is no $\|A\|$
- Then, if $x_n \to x$ then perhaps $Ax_n \nrightarrow Ax$
- One must be aware of $\mathcal{D}(A)$

Partial Diff. Eqs. $\rightarrow$ Ordinary ∞-dim. Diff. Eqs.

Most popular methods:

- Solutions using semigroups of operators, mild solutions to semiliear equations (Peszat, Zabczyk).
- Soultions in multi-Hilbertian spaces, e.g. in the dual to a nuclear space (Itô, Kallianpur).
- Variational solutions in a Gelfand triplet (Agmon, Lions, Röckner).
- Solutions using Dirichlet forms (Albeverio, Osada (Itô's prize in 2013))
- White noise method (Hida)
- Brownian sheet method (Walsh)
- Solutions in R^∞ (Leha, Ritter)
- Rough paths solutions (Friz, Hairer)

Partial Diff. Eqs. $\to$ Ordinary ∞-dim. Diff. Eqs.

Example (Heat Equation $\to$ Abstract Cauchy Problem)

- One-dimensional Heat Equation,

$$\begin{cases} u_t(t,x) & = u_{xx}(t,x),\ t>0 \\ u(0,x) & = \varphi(x),\ x\in\mathbb{R} \end{cases} \Rightarrow \begin{cases} \frac{du(t)}{dt} & = \Delta u(t),\ t>0 \\ u(0) & = \varphi \in X \end{cases}$$

Heat Equation is converted into an Abstract Cauchy Problem in a Banach space X of bounded, uniformly continuous functions.
(Differentiation (in t) is in the sense of Banach space.)

Why Semigroups of Operators?

The uniqueness of solutions and the Hughense principle imply that temperature $u^{\varphi}(t,x)$ can be calculated for the initial condition φ, but also from an intermediate value $u^{\varphi}(s,x)$, thus

$$u^{\varphi}(t,x) = \left(G(t)\varphi\right)(x) = G(t-s)u^{\varphi}(s,x) = \left(G(t-s)G(s)\varphi\right)(x).$$

and we have a semigroup of operators $G(t)$. For Δ, it is a Gaussian semigroup on the Hilbert space $L^2\left(\mathbb{R}\right)$

$$\left(G(t)\varphi\right)(x) = \begin{cases} \dfrac{1}{(4\pi t)^{1/2}} \displaystyle\int_{\mathbb{R}} \exp\left\{-|x-y|^2/4t\right\} \varphi(y)\,dy, & t>0 \\ \varphi(x), & t=0. \end{cases}$$

The infinitesimal generator of this C_0 semigroup of contractions is

$$\Delta = \frac{d^2}{dx^2}$$

whose domain is

$$\mathcal{D}(\Delta) = W^{2,2}(\mathbb{R})$$

What can be done if the Peano Theorem fails?

In the case of semilinear equations

$$\frac{du}{dt}(t) = Au(t) + G(u(t)), \quad u(0) = x \in H$$

we can salvage the Peano theorem if, for example, the semigroup of operators generated by A is compact (that is, sets $S(t)(Boundedset)$ have compact closures).

Time dependence

An important observation is that, because the operator A does not depend on time, the operator $T(t)$ transforming the solution $u(s)$ from time s to $u(t+s)$ at time $(t+s)$ does not depend on s.

The importance of this is that the physical laws governing the process described by the equation are time independent.

Mild Solutions

In the presence of an external cooling or heating

$$\begin{cases} \frac{du(t)}{dt} & = \Delta u(t) + f(t),\ t > 0 \\ u(0) & = \varphi \in X \end{cases} \Rightarrow \begin{cases} \frac{du(t)}{dt} & = Au(t) + f(t),\ t > 0 \\ u(0) & = \varphi \in X \end{cases}$$

a differentiable solution may not exist, even if the initial condition $\varphi \in \mathcal{D}(A)$ and the forcing function f is continuous. The mild solution:

$$u(t) = S(t)\varphi + \int_0^t S(t-s)f(s)\,ds$$

is a classical solution (continuously differentiable and in $\mathcal{D}(A)$) if, for example, $\varphi \in \mathcal{D}(A)$ and the convolution is differentiable (we will see this later)...

...but, for example, even if the initial condition $u(0) = 0 \in \mathcal{D}(A)$ and

$$f(t) = S(t)x, \quad \text{for } x \in H, \ x \notin \mathcal{D}(A),$$

then

$$\begin{aligned} u(t) &= S(t)0 + \int_0^t S(t-s)f(s)\,ds = \int_0^t S(t-s)S(s)x\,ds \\ &= \int_0^t S(t)x\,ds = tS(t)x \end{aligned}$$

which is not differentiable as

$$\lim_{h\to 0} \frac{(t+h)S(t+h)x - tS(t)x}{h} = tS(t)\lim_{h\to 0} \frac{S(h)x - x}{h} + S(t)x$$

and the limit d.n.e.

How can we interpret mild solutions?

Theorem (Meta-theorem (Peszat, Zabczyk))

Class of weak solutions (in the sense of evaluation on a test function) = Class of mild solutions.

Definition

A function $u(t)$ is a weak solution to the Abstract Cauchy Problem in a Hilbert space H (e.g. $W^{1,2}(\mathbb{R}) \subset L_2$), if for every $h \in \mathcal{D}(A^*)$

$$\langle u(t), h\rangle_H = \langle u(0), h\rangle_H + \int_0^t \left(\langle u(s), A^*h\rangle_H + \langle f(s), h\rangle_H\right) ds$$

First, we can interpret weak solutions as values not at a single point but as values on neighbourhoods of points.

One can think of a mild solution $u(t)$ as a weak solution of the heat equation, that is as of a thermometer (mercury column) and the test function h is thermometer's shape.

Isn't it how we (used to) measure temperature?

Stochastic Equations

The equation is on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$

$$
\begin{aligned}
dX(t) &= (AX(t) + F(X(t)))\,dt + B(X(t))\,dW_t \\
&+ \int_{H\setminus\{0\}} f(v, X(t))\,q(dt, dv) \qquad (4.1) \\
X(0) &= \xi,
\end{aligned}
$$

K, H - are real separable Hilbert spaces.
β is a Lévy measure
$q(ds, dv) = N(ds, dv)(\omega) - ds\beta(dv)$ - compensated Poisson measure
$N(ds, dv)(\omega)$ σ - finite Poisson measure on σ-field $\mathcal{B}(\mathbb{R}_+ \times H \setminus \{0\})$
$(W_t)_{t\leq T}$ - Q-Wiener process in K
A generates C_0-semigroup $\{S(t), t \geq 0\}$ on H,
Suitable F, B and f .

Mild Solutions

Definition

A stochastic process $\{X(t), t \leq T\}$ na $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$ is a mild solution to (4.1) on $[0,T]$, if for $0 \leq t \leq T$

(i) $X(t)$ is $\mathcal{F}_t$-measurable,

(ii) $\{X(t), t \leq T\}$ is jointly measurable and $\int_0^T E\,\|X(t)\|_H^2 \,dt < \infty$,

(iii) na $[0,T]$ P-a.s.

$$X(t) = S(t)\xi + \int_0^t S(t-s)F(X(s))\,ds + \int_0^t S(t-s)B(X(s))\,dW_s + \int_0^t \int_{H\setminus\{0\}} S(t-s)f(v,X(s))\,q(ds,dv)$$

Exist under the assumptions of linear growth and Liptschiz conditions, ξ can $\notin \mathcal{D}(A)$ but needs to be square integrable.

Strong Solutions

Definition

A stochastic process $\{X(t), t \leq T\}$ on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$ is a strong solution to equation (4.1) on $[0, T]$, if for $0 \leq t \leq T$
(i) $X(t)$ is an $\mathcal{F}_t$-measurable process, càdlàg (right continuous with left limits),
(ii) $X(t) \in \mathcal{D}(A)$, $dt \otimes dP$-p.n. and $\int_0^T \|AX(t)\|_H^2 \, dt < \infty$, P-a.s.
(iii) on $[0, T]$ P-a.s.

$$
\begin{aligned}
X(t) = \xi + \int_0^t \left(AX(s) + F(X(s))\right) \, ds + \int_0^t B(X(s)) \, dW_s \\
+ \int_0^t \int_{H \setminus \{0\}} f(v, X(s)) \, q(ds, dv)
\end{aligned}
$$

Mild solutions are strong solutions if $X(t) \in \mathcal{D}(A)$.

Itô Formula.

One-dimensional case:

$$\begin{aligned} X(t) &= X(0) + \int_0^t \Psi(s)\,ds + \int_0^t \Phi(s)\,dW_s \\ \mathcal{H}(t, X(t)) &= \mathcal{H}\left(0, X(0)\right) + \int_0^t \mathcal{H}_x\left(s, X(s)\right)\Phi(s)\,dW_s \\ &\quad + \int_0^t \Big\{\mathcal{H}_t\left(s, X(s)\right) + \mathcal{H}_x\left(s, X(s)\right)\Psi(s) \\ &\quad + \frac{1}{2}\mathcal{H}_{xx}\left(s, X(s)\right)\Phi^2(s)\Big\}\,ds \end{aligned}$$

by Taylor's formula

$$\begin{aligned} d\mathcal{H}\left(t, X(t)\right) &= \mathcal{H}_t\left(t, X(t)\right)\,dt + \mathcal{H}_x\left(t, X(t)\right)\,\left(\Psi(t)\,dt + \Phi(t)\,dW_t\right) \\ &\quad + \frac{1}{2}\mathcal{H}_{xx}\left(t, X(t)\right)\left(\Psi(t)\,dt + \Phi(t)\,dW_t\right)^2 + \ldots \end{aligned}$$

Itô formula.

Multidimensional and infinitely dimensional cases:

$$
\begin{aligned}
X(t) &= X(0) + \int_0^t \Psi(s)\,ds + \int_0^t \Phi(s)\,dW_s \\
\mathcal{H}(t, X(t)) &= \mathcal{H}(0, X(0)) + \int_0^t \langle \mathcal{H}_x(s, X(s)), \Phi(s)\,dW_s \rangle_H \\
&\quad + \int_0^t \Big\{ \mathcal{H}_t(s, X(s)) + \langle \mathcal{H}_x(s, X(s)), \Psi(s) \rangle_H \\
&\quad + \frac{1}{2} \operatorname{tr}\Big[\mathcal{H}_{xx}(s, X(s)) \left(\Phi(s) Q^{1/2}\right) \left(\Phi(s) Q^{1/2}\right)^* \Big] \Big\}\, ds \\
&= \mathcal{H}(0, X(0)) + \int_0^t \langle \mathcal{H}_x(s, X(s)), \Phi(s)\,dW_s \rangle_H \\
&\quad + \int_0^t \Big\{ \mathcal{H}_t(s, X(s)) + \mathcal{L}\mathcal{H}(s, X(s)) \Big\}\, ds
\end{aligned}
$$

multiplication is replaced by a scalar product and the square by a trace of an operator composed with its adjoint (Q is the covariance operator).

Itô formula.

Compensated Poisson measure - infinite dimensional case (specific case, useful for solutions of SDE's):

$$
\begin{aligned}
X(t) &= \int_0^t \int_{H\setminus\{0\}} f(v,s)q(ds\,dv) \\
\mathcal{H}(t,X(t)) &= \mathcal{H}\left(0,X(0)\right) + \int_0^t \mathcal{H}_t(s,X(s))\,ds \\
&+ \int_0^t \int_{H\setminus\{0\}} \Big\{\mathcal{H}\left(s,X(s)+f(v,s)\right) \\
&\qquad -\mathcal{H}(s,X(s))\Big\} q(dv\,ds) \\
&+ \int_0^t \int_{H\setminus\{0\}} \Big\{\mathcal{H}(s,X(s)+f(v,s)) - \mathcal{H}(s,X(s)) \\
&\qquad -\langle \mathcal{H}_x(s,X(s)), f(v,s)\rangle_H \Big\}\,\beta(dv)\,ds
\end{aligned}
$$

Itô Formula for Strong Solutions.

$$\begin{aligned}\mathcal{H}(t,X(t)) &= \mathcal{H}(0,X(0)) + \int_0^t \left(\mathcal{H}_s(s,X(s)) + \mathcal{L}\mathcal{H}(s,X(s))\right) ds \\ &\quad + \int_0^t \langle \mathcal{H}_x(s,X(s)), B(X(s))\, dW_s\rangle_H \\ &\quad + \int_0^t \int_{H\setminus\{0\}} \Big[\mathcal{H}(s,X(s)+f(v,X(s))) \\ &\qquad\qquad - \mathcal{H}(s,X(s))\Big]\, q(dv,ds),\end{aligned}$$

$$\begin{aligned}\mathcal{L}\mathcal{H}(s,x) &= \langle \mathcal{H}_x(s,x), Ax + F(x)\rangle_H \\ &\quad + \frac{1}{2}\mathrm{tr}(\mathcal{H}_{xx}(s,x)B(x)Q(B(x))^*) \\ &\quad + \int_{H\setminus\{0\}} \Big[\mathcal{H}(s,x+f(v,x)) - \mathcal{H}(s,x) \\ &\qquad\qquad - \langle \mathcal{H}_x(s,x), f(v,x)\rangle_H\Big]\, \beta(dv).\end{aligned}$$

Approximating Mild Solutions

For $n \in \rho(A)$ (the resolvent set), $R(n,A) = (nI - A)^{-1}$ is the resolvent,

$$A_n = AnR(n,A) \in L(H) \quad \text{Yosida approximation of the operator } A$$

Formally: $A\frac{n}{n-A} = A\frac{1}{1-A/n} \to A$, but only:

$$\lim_{n\to\infty} nR(n,A)x = x,\ x \in H, \quad \text{and} \quad \lim_{n\to\infty} A_n x = Ax,\ x \in \mathcal{D}$$

One method: replace an unbounded with a bounded operator and keep the initial condition in H and use the same coefficients

$$\begin{aligned} dX_n(t) &= (A_n X_n(t) + F(X_n(t)))\,dt + B(X_n(t))\,dW_t \\ &+ \int_{H\setminus\{0\}} f(v, X_n(t))\,q(dt,dv) \\ X(0) &= \xi \in H \end{aligned}$$

Approximating Mild Solutions

Another method: Change the initial condition but keep the unbounded operator A and use $R_n = nR(n, A) \xrightarrow{formally} I$

$$\begin{aligned} dX_n(t) &= (AX_n(t) + R_n F(X_n(t)))\,dt + R_n B(X_n(t))\,dW_t \\ &+ \int_{H\setminus\{0\}} R_n f(v, X_n(t))\,q(dt, dv) \\ X(0) &= \xi \in \mathcal{D}(A) \end{aligned}$$

In both cases the solution is strong strong, as either the operator is bounded or all convolutions are in $\mathcal{D}(A)$.

If we keep A, the counterpart of $\mathcal{L}$ is

$$\begin{aligned} \mathcal{L}_n \mathcal{H}(s, x) &= \langle \mathcal{H}_x(s, x), Ax + R_n F(x) \rangle_H \\ &+ \frac{1}{2}\mathrm{tr}(\mathcal{H}_{xx}(s, x) R_n B(x) Q (R_n B(x))^*) \\ &+ \int_{H\setminus\{0\}} \Big[\mathcal{H}(s, x + R_n f(v, x)) - \mathcal{H}(s, x) \\ &\quad - \langle \mathcal{H}_x(s, x), R_n f(v, x) \rangle_H \Big] \beta(dv). \end{aligned}$$

Itô Formula (or Method)for Mild Solutions.

$$\lim_{n\to\infty}\int_0^t \langle \mathcal{H}_x(s,X_n(s)), AX_n(s)\rangle_H \, ds = \mathcal{H}(t,X(t)) - \mathcal{H}(0,X(0))$$

$$-\int_0^t \mathcal{H}_s(s,X(s))\,ds - \int_0^t \langle \mathcal{H}_x(s,X(s)), F(X(s))\rangle_H \, ds$$

$$-\int_0^t \frac{1}{2} tr(\mathcal{H}_{xx}(s,X(s))(B(X(s)))Q(B(X(s)))^*)\,ds$$

$$-\int_0^t \int_{H\setminus\{0\}} \Big[\mathcal{H}(s,X(s)+f(v,X(s))) - \mathcal{H}(s,X(s))$$

$$-\langle \mathcal{H}_x(s,X(s)), f(v,X(s))\rangle_H \Big] \beta(dv)ds$$

$$-\int_0^t \langle \mathcal{H}_x(s,X(s)), B(X(s))\,dW_s\rangle_H$$

$$-\int_0^t \int_{H\setminus\{0\}} [\mathcal{H}(s,X(s)+f(v,X(s))) - \mathcal{H}(s,X(s))]\; q(dv,ds).$$

Itô Method for Mild Solutions.

From the proof, we obtain this important property (we will use this soon)

$$\lim_{n\to\infty} \left|\mathcal{L}\mathcal{H}\left(t, X_n(t)\right) - \mathcal{L}_n\mathcal{H}\left(t, X_n(t)\right)\right| = 0, \quad P - a.s.$$

$\mathcal{L}$ contains the unbounded operator A (but is evaluated in $X_n(t) \in \mathcal{D}(A)$, and $\mathcal{L}_n$ uses R_n.

Itô Method for Mild Solutions.

Exponential stability in the mean square sense.

$$E\,\|X^x(t)\|_H^2 \le ce^{-\gamma t}\|x\|_H^2, \quad c,\gamma > 0$$

Meta-Theorem: For mild solutions, existence of a Lapunov function is a sufficient condition for this form of stability

$$\Lambda : H \to \mathbb{R} \in C^2(H), \quad t \ge 0,\ x \in H$$

$$c_1\|x\|_H^2 \le \Lambda(x) \le c_2\|x\|_H^2, \quad c_1, c_2 > 0$$

$$\mathcal{L}\Lambda(x) \le -c_3\Lambda(x), \quad \text{for all } \ x \in \mathcal{D}(A),\ c_3 > 0.$$

Itô-Yosida Method for Mild Solutions.

Proof

We use the Itô formula for Yosida approximations (strong solutions) $X_n(t)$ and function $e^{c_3 t}\Lambda(x)$ and apply the expected value. For the initial condition $x \in \mathcal{D}(A)$

$$e^{c_3 t} E\Lambda\left(X_n(t)\right) - \Lambda\left(x\right) = E\int_0^t e^{c_3 s}\left(c_3\Lambda\left(X_n(s)\right) + \mathcal{L}_n\Lambda\left(X_n(s)\right)\right) ds$$

From the properties of Lapunov function

$$c_3\Lambda\left(X_n(s)\right) + \mathcal{L}_n\Lambda\left(X_n(s)\right) \leq -\mathcal{L}\Lambda\left(X_n(s)\right) + \mathcal{L}_n\Lambda\left(X_n(s)\right)$$

hence

$$e^{c_3 t} E\Lambda\left(X_n(t)\right) - \Lambda\left(X_n(0)\right) \leq E\int_0^t e^{c_3 s}\left(-\mathcal{L}\Lambda\left(X_n(s)\right) + \mathcal{L}_n\Lambda\left(X_n(s)\right)\right) ds$$

from the Itô method, the integral goes to zero, thus

$$E\Lambda(X(t)) \leftarrow E\Lambda(X_n(t)) \leq e^{-c_3 t}\Lambda(x)$$

and simple manipulations lead to stability.

Itô-Yosida Method for Mild Solutions.

Exponential ultimate boundedness in the mean square sense.

$$E\,\|X^x(t)\|_H^2 \le ce^{-\gamma t}\|x\|_H^2 + M, \quad c,\gamma,M>0$$

For mild solutions, it suffices that a Lapunov function exists

$$\Lambda : H \to \mathbb{R} \in C^2(H), \quad t \ge 0,\ x \in H$$

$$c_1\|x\|_H^2 - k_1 \le \Lambda(x) \le c_2\|x\|_H^2 - k_2, \quad c_1,c_2,k_1,k_2>0$$

$$\mathcal{L}\Lambda(x) \le -c_3\Lambda(x) + k_3, \quad \text{for all } \ x \in \mathcal{D}(A),\ c_3,k_3>0.$$

Similar proof as of exponential stability.

Example - Stochastic Heat Equation

$$dX(x,t) = \frac{\partial^2}{\partial x^2} X(x,t)\,dt + \sigma X(x,t)\,dW_t + \int_{H\setminus\{0\}} f(v) X(x,t)\,q(dv,dt).$$

$x \in (0,1),\ X(0,t) = X(1,t) = 0,\ X(x,0) = X_0(x) \in L_2(0,1);$

$B(x) = \sigma x,\ A = \frac{d^2}{dx^2},\ f \in L_2(0,1).$

$H = L_2(0,1), \mathcal{D}(A) = \{g \in H \mid g',\, g'' \in H,\ g(0) = g(1) = 0\}.$

The solution is exponentially stable

$$E\|X(t)\|^2 \leq \lambda e^{-kt} \|X(0)\|^2$$

Invariant Measure

A mild solutions to a SDE is a Markov process, and generates a transition semigroup

$$(P_t\varphi)(x) = E(\varphi(X^x(t))), \quad x \text{ - initial condition}$$

If a mild solution is ultimately bounded then there exists an invariant measure

$$\mu := \mu(A) = \int_H P_t(x, A)\mu(dx)$$

Other Itô formulas.

For a bounded operator A we get the usual Itô formula.

Itô-Ichikawa formula requires an extension of $\mathcal{LH}(t,x)$ to a continuous function $\overline{\mathcal{LH}}(t,X(s))$ na $[0,T]\times H$. It may take the form

$$\begin{aligned}
&\overline{\mathcal{LH}}(t,x) = \\
&\lim_{n\to\infty} \langle \mathcal{H}_x(s,X_n(s)), AX_n(s)\rangle_H \\
&+\langle \mathcal{H}_x(s,X(s)), F(X(s))\rangle_H \\
&+\frac{1}{2} tr(\mathcal{H}_{xx}(s,X(s))(B(X(s)))Q(B(X(s)))^*) \\
&+\int_{H\setminus\{0\}} \Big[\mathcal{H}(s,X(s)+f(v,X(s))) - \mathcal{H}(s,X(s)) \\
&-\langle \mathcal{H}_x(s,X(s)), f(v,X(s))\rangle_H \Big]\beta(dv)
\end{aligned}$$

Other Itô formulas.

If $\mathcal{H}(t,x) = e^{lA}\Lambda(t,x)$ i $\Lambda(t,x) \in \mathcal{D}(A)$,

$$\begin{aligned}\lim_{n\to\infty} \langle \mathcal{H}_x(s, X_n(s)), AX_n(s)\rangle_H &= \left\langle A^* e^{lA}\Lambda_x(t, X_n(s)), X_n(s)\right\rangle_H \\ &\to \left\langle A^* e^{lA}\Lambda_x(t, X(s)), X(s)\right\rangle_H\end{aligned}$$

Oter Itô formulas.

DaPrato, Jentzen and Röckner obtained Itô formula in a semigroup form
It agrees with other Itô formulas discussed here in the case of

$$\mathcal{H}(t,x) = e^{(t-s)A}\Gamma(x).$$

Closing Comments

- I thank the organizers for providing this opportunity
- There was rich research collaboration among Professor Mandrekar's Students (Milan Merkle (1984), Philip Richard (1990), Sixiang Zhang (1991), Ruifeng Liu (1997), Philip Gerrish (1998), Juan Du (2007)
- A new lead - student research project inspired by Philip Gerrish: "Are teams of misfits successful?"